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UNIVERSITY COLLEGE, UNIVERSITY OF LONDON.

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TECHNICAL SERIES, II.

ON SOME DISREGARDED POINTS IN THE STABILITY OF MASONRY DAMS.

BY

L. W. ATCHERLEY,
DEMONSTRATOR IN APPLIED MECHANICS, UNIVERSITY COLLEGE, LONDON,

WITH SOME ASSISTANCE FROM

KARL PEARSON, F.R.S.,
PROFESSOR OF APPLIED MATHEMATICS AND MECHANICS, UNIVERSITY COLLEGE, LONDON.

[WITH ONE FIGURE IN THE TEXT AND THREE PLATES.]

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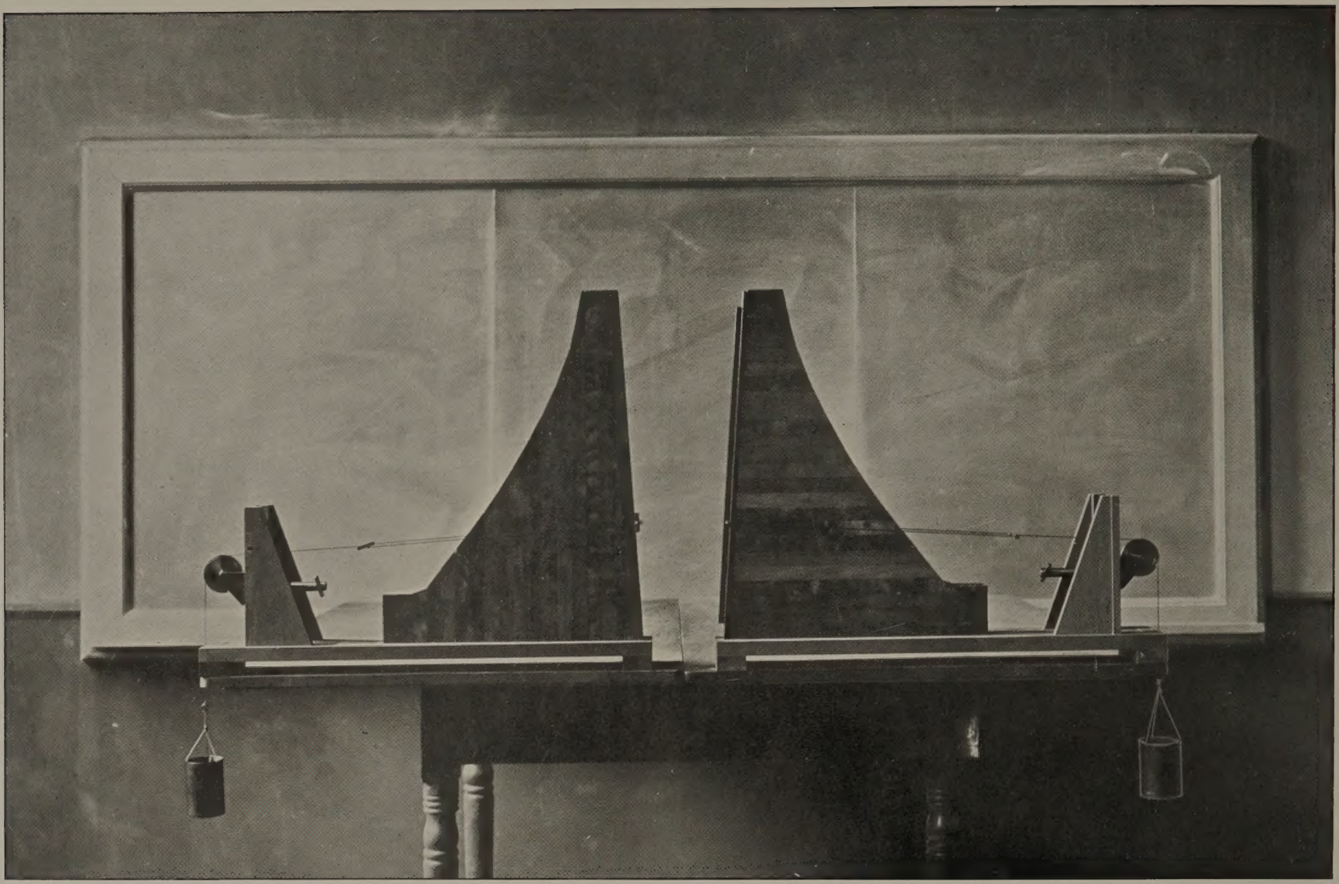
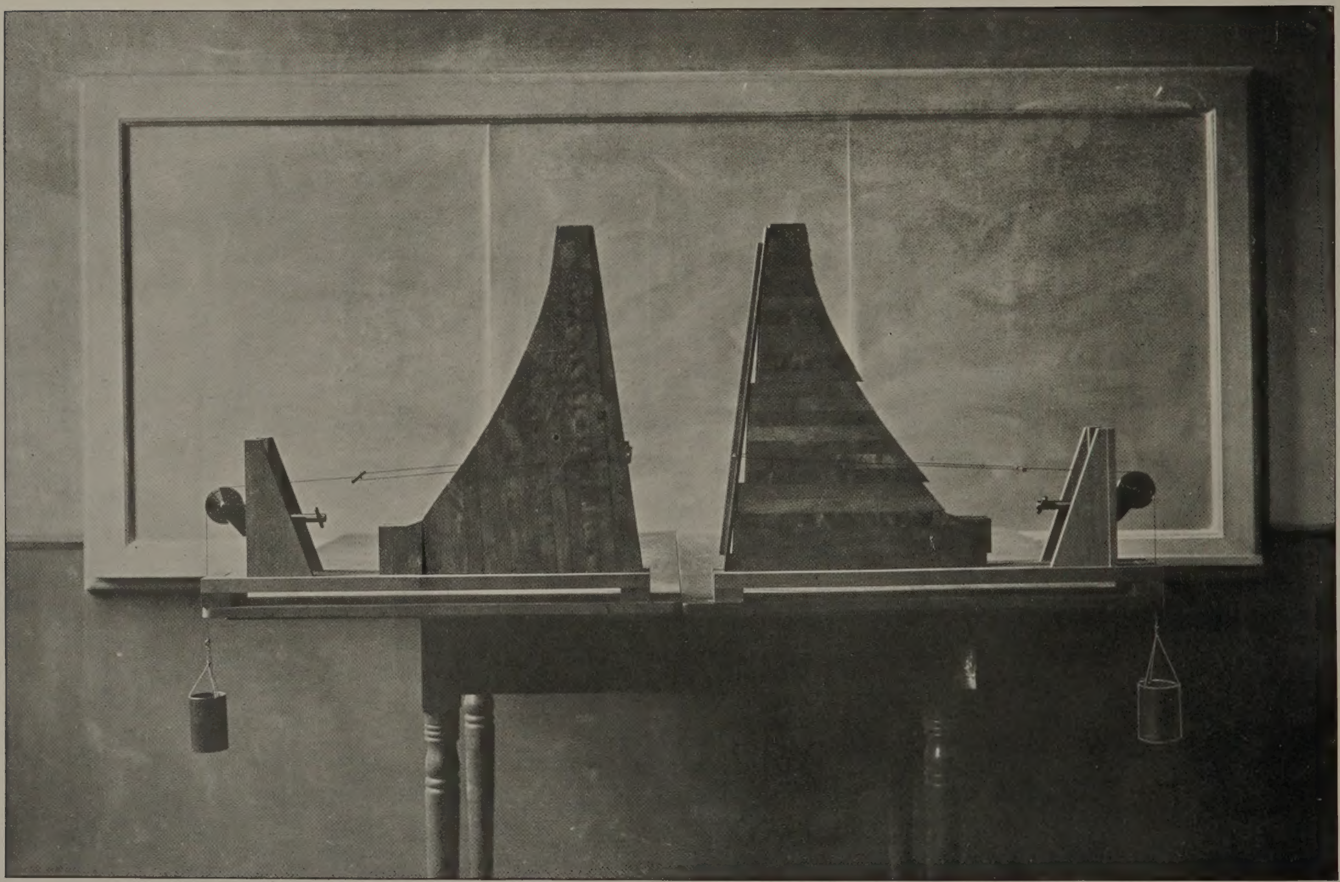
Plate I. Model Dams.



(i.) Before Water Pressure applied.



(ii.) Collapsing under Water Pressure.

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On some Disregarded Points in the Stability of Masonry Dams.

By L. W. ATCHERLEY, *Demonstrator in Applied Mechanics, University College, London.*

With some Assistance from KARL PEARSON, *F.R.S., Professor of Applied Mathematics and Mechanics, University College, London.*

(1.) *Introductory.*

THE most cursory examination of any of the current practical text-books suffices to show the enquirer that the present drawing office treatment of dam construction is defective and needs careful revision and extension.

It is well known that owing to (i.) the manner in which masonry structures are built, the mixture of stones and cement, and (ii.) the nature of the sections at the springings or areas of support, it is extremely difficult, if not impossible, to apply to them the general theory of elastic bodies. Accordingly a theory has been developed of the following nature. We take a series of sections of the masonry structure; in the case of the arch, these sections are perpendicular to the *central line* of the arch, which in arches is more or less clearly defined by the fact that the transverse dimensions from extrados to intrados are usually small as compared with the length of the arch, and we can speak more or less correctly of a true *cross-section.* In dams it has been usual to take the sections horizontally, and these can only be considered *cross-sections* by courtesy, for they are not invariably, or even usually, perpendicular to the *central line,* which may be defined as the line joining their centroids; such perpendicularity is a feature of the arch, but not of the dam. Now, if such sections, real or pseudo-cross-sections, be taken, and the resultant formed of all the external forces—loads, self-weight, reactions—for the structure on one side of the cross-section, this resultant strikes the cross-section in a point termed the *load-point* of that cross-section. If the load-points of the successive cross-sections be joined up we obtain a polygon or curve, which in the case of an arch is termed its *line of pressure,* in the case of a dam its *line of resistance.* The chief difference between lines of pressure and lines of resistance is that in the former case the line meets the section not only in the load-point, but indicates by the direction of its tangent the direction of the resultant force on the section; in the latter case the line of resistance gives the point of application of the resultant force, but its tangent is not the direction of that resultant. We thus distinguish, therefore, load-point loci of two classes, the line of pressure of the arch and the line of resistance of the dam.

If the resultant force on any cross-section be resolved along and perpendicular to the cross-section, the component along the cross-section is termed the *total shear on the cross-section*, and the component perpendicular to the cross-section the *total thrust on the cross-section.* If the loads and reactions on the structure lie in one plane, and this plane cut the cross-sections in a principal axis of inertia of each cross-section, as is usual in the case of arches and dams, then the moment of the thrust about a line, the *central axis*, through the centroid of the cross-section in its plane perpendicular to the above-mentioned principal axis is the *bending moment.** From a knowledge of *total shear*, *total thrust*, and either *load-point* or *bending moment*, we must, by adopting some hypothesis or other, be enabled to deduce the maximum stresses at any point of the cross-section. If the loading be symmetrical with regard to the cross-section we shall not be troubled with any torsional effect. The chief shearing stresses over the cross-section will be parallel to the line joining centroid to load-point, and the tensile and compressive stresses will be distributed symmetrically about this line. It is usual to assume that the latter stresses are proportional to their distances from a certain straight line termed the *neutral axis;* this axis is perpendicular to the line joining the load-point to the centroid (conjugate to it in direction in the more general case), and is so situated that the *spin radius* (*radius of gyration*) of the area of the section about its central line is a geometrical mean between the distances of the load-point and neutral axis from the centroid. Further, the resultant normal stress on the cross-section, which must be equal and opposite to the resultant external force resolved perpendicular to the cross-section, *i.e.*, the thrust, is equal to the area of the cross-section into the stress at the centroid. The nature of the distribution of normal stresses thus assumed would be closely reached in a column of uniform cross-section equal to the given cross-section of the masonry structure at any selected point and having its terminal section loaded with the same total thrust at the same load-point. The very gradual change of the cross-section in the case of an arch of the usual type renders this hypothesis more reasonable in the case of an arch than it is in the case of a dam. Probably, however, it does not deviate widely from the truth even in the case of a masonry dam with the usual slight batter on the front or fluid side, except for points quite close to the curved or polygonal surface of the flank. RANKINE has endeavoured to allow for a change in the stresses at the flank, but his hypothesis seems without real justification, and we should prefer merely to say that the above distribution of stress should not be considered as absolutely holding right up to the flank of the dam. Of the general principles involved in the above assumptions RANKINE himself writes* :—"I have not attempted to deduce the ratio which these quantities (pressures at faces of dam) should bear to one another from the theory of the distribution of the stress in a solid body; for the data on which any such theoretical determination would have to be based are too uncertain." But as he strives to get his lines of pressure for full and empty reservoir inside the "middle third," he really is assuming that very distribution of stress upon the cross-section which he directly repudiates. The same tacit assumption occurs throughout RANKINE's treatment of the subject in his 'Applied Mechanics,' Arts. 216–219.

---

* If the above conditions be not fulfilled, the bending moment is taken about the conjugate to the line joining the centroid to the load-point with regard to the momental ellipse of the section. This more general definition, of importance in some metal structures, is not needful in the case of arches and dams.

In the case of most arches and dams the cross-section or strip of area we have to deal with is rectangular. Hence, if it be of depth ζ, perpendicular to the neutral axis, and the load-point be at distance c from the centroid, the neutral axis at distance d, we have

$$c \times d = \tfrac{1}{12}\zeta^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{i.}).$$

Further

$$C_{\text{max.}} = \frac{Q}{A}\left(1 + \frac{6c}{\zeta}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{ii.}).$$

$$T_{\text{max.}} = \frac{Q}{A}\left(\frac{6c}{\zeta} - 1\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{iii.}),$$

where Q is the total thrust, A the area of the cross-section and $C_{\text{max.}}$, $T_{\text{max.}}$, the maximum compressive and tensile stresses in the cross-section.

It has not hitherto been usual in the treatment of dams to consider the distribution of the shearing stresses on the section. But, whatever be the validity of the distribution of normal stresses assumed above, it follows for the usual types of dam construction with almost equal validity that the shear S at distance y from the central axis is given by

$$S = \tfrac{3}{2}\frac{P}{A}\left(1 - \frac{4y^2}{\zeta^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{iv.}),$$

where P is the total shear on the cross-section.† If we take a vertical transverse section of the dam at a considerable distance from either end, there is no more reason why the material on the right of such section should at any point shear in the one direction more than in the other the material on the left, and consequently, except in the neighbourhood of the terminal sections, we may put all shears on transverse vertical sections zero. This must be absolutely true for the mid-section of a symmetrical dam, and amounts to the assumption that we consider any portion of the dam between two vertical transverse sections as supporting the fluid pressure and its own weight by the forces between the substratum and the dam itself. This is, in fact, the usual assumption made when in dealing with the dam we neglect the supporting forces at the terminals—the sort of arch or beam action of the dam as a whole—as compared with the actions of the substratum on the base of the dam. Even if the effect of these terminal actions be not neglected, yet, at some little distance from the terminal, the distribution of stress over two vertical transverse sections at small distance from each other must be nearly the same, and their difference produce insensible effect on the shear across the horizontal sections.

---

* RANKINE, 'Miscellaneous Scientific Papers,' p. 552.

† See the consideration of this point in the Appendix to this memoir.

Now equations (i.) to (iv.) seem to us far more reasonable than the usual dam conditions of equilibrium, which tacitly assume (i.) to (iii.) when they impose the condition of the line of resistance falling within the middle third, and then use a friction condition instead of a shear condition for (iv.). Thus M. LÉVY writes :*

> " *Conditions de résistance d'un mur de soutènement.*
>
> " Si un mur est soumis à l'action de son propre poids et à des forces horizontales ou inclinées il faut, comme pour les pieds-droits d'une voûte, que la résultante de la pression exercée sur le pied du mur et généralement sur toute assise horizontale :—
>
> " (1) Coupe cette assise dans son tiers moyen ;
>
> " (2) Fasse avec la verticale un angle moindre que l'angle du frottement de la maçonnerie sur elle-même (on admet généralement que cet angle peut aller jusqu'à 30 degrés).
>
> " (3) Que la pression maxima par unité de surface ne dépasse pas une valeur donnée dépendant de la nature de la pierre employée."

(3) of course would be generally accepted, but the only way of finding the maximum pressure is that indicated in the preamble " comme pour les pieds-droits d'une voûte," and tacitly assumed in (1) (see p. 552 of LÉVY'S work),† *i.e.*, the assumption involved in our equations (i.) to (iii.) above.

We now turn to the frictional condition involved in (2) above. RANKINE and some writers give even 36 degrees instead of LÉVY'S safer 30 degrees for the angle of friction. Now the difficulty of this frictional condition to our minds is, (1) that it leaves us in absolute doubt as to the actual distribution of shearing stress over any cross-section, and (2) that in actually constructed dams we have found the angle of friction certainly to exceed 30 degrees. Some measure of this shearing stress seems to us all important, for without it we cannot determine the tensile and compressive stresses across *the vertical sections of the tail of the dam.* We see no reason

---

[*] 'La Statique graphique,' IV^e Partie, " Ouvrages en Maçonnerie," p. 92. These conditions are applied to dams with battered face and curved flank (see §§ 601 and 603). Our appendix discusses a later algebraic treatment by LÉVY.

[†] M. FLAMMANT uses the same relation (see 'Encyclopédie des Travaux Publics,' " Stabilité des Constructions," p. 82). He, however, admits that " Lorsque les maçonneries son bien faites, elles peuvent réellement supporter de petits efforts d'extension," p. 83.

whatever why dams should be tested solely by taking horizontal cross-sections, and asserting that the line of resistance must lie in the horizontal middle third, and absolutely neglecting the stresses across the vertical sections of the tail. We consider that if the former condition is valid, then no dam ought to be passed unless it can be shown that there is no tension of any serious value across the vertical cross-sections of the tail. We believe that a great number of dams as now designed will be found to have very substantial tension in the tail, and this is in our opinion a source of weakness in dam construction which has not been properly considered and allowed for.

In order to deal with the stresses in the vertical cross-sections of the dam, it is necessary to make some hypothesis as to the distribution of shearing stress over the base of the dam, as well as the distribution of pressures already assumed, openly or tacitly, by previous writers.

Now (i.) to (iii.) above would only be closely true if the dam were a thick wall with vertical front and flank; but, under these circumstances, (iv.) would be equally true.* Hence there is no less and no greater assumption made when we extend (iv.) as well as (i.) to (iii.) to dams with battered faces and curved flanks. But, as we shall show in the sequel, the exact form of (iv.) is probably not of very great importance. If we assume the shear to be actually distributed uniformly over the base—an extreme and quite impossibly favourable case—we still find the tail of the dam to be in considerable tension. It is usual to thicken up the tail at its tip, probably on the view that the collapse of the dam might arise from the shearing off of the tip of the tail. But if dams have collapsed from tail failure, we shrewdly suspect that this has arisen first, not from the shearing, but from the tensional stresses in the tail. Anyhow, this strengthening of the tail extends such a little way in most dams of which we have examined the profiles (while the line of resistance may not enter the middle third of the vertical sections till we get to nearly the mid-vertical section of the dam) that it has sensibly no effect on the element of weakness which we are now endeavouring to indicate.

(2.) If we take the general equations of elasticity in the notation adopted in TODHUNTER and PEARSON'S 'History of Elasticity'; considering the axis of z as vertically downwards, the axis of x perpendicular to it in the plane of the profile, and y perpendicular to both along the length of the dam, then the shift v parallel to y is zero, and our body stress equations become, since no stress is a function of y,

$$\left.\begin{aligned}\frac{d\widehat{xx}}{dx}+\frac{d\widehat{xz}}{dz}&=0\\ \frac{d\widehat{xz}}{dx}+\frac{d\widehat{zz}}{dz}+\rho g&=0\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(\text{v.})$$

---

* See Appendix.

where ρ is the density of the masonry, the second body-stress equation being identically satisfied.

These give us

$$\left.\begin{aligned} \widehat{zz} &= \frac{d^2 V}{dx^2} - g\rho z \\ \widehat{xx} &= \frac{d^2 V}{dz^2} \\ \widehat{xz} &= -\frac{d^2 V}{dx\,dz} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vi.)},$$

V being at present an arbitrary function.

If we suppose *average* isotropy of material,

$$\widehat{xx} = \lambda(s_x + s_z) + 2\mu s_x, \qquad \widehat{zz} = \lambda(s_x + s_z) + 2\mu s_z \,. \quad . \quad . \quad . \quad \text{(vii.)}$$

Or

$$2s_x = \frac{\widehat{xx} + \widehat{zz}}{\lambda + 2\mu} + \frac{\widehat{xx} - \widehat{zz}}{2\mu} = 2\frac{du}{dx},$$

$$2s_z = \frac{\widehat{xx} + \widehat{zz}}{\lambda + 2\mu} - \frac{\widehat{xx} - \widehat{zz}}{2\mu} = 2\frac{dw}{dz}.$$

But

$$\frac{du}{dz} + \frac{dw}{dx} = \frac{1}{\mu}\widehat{xz},$$

therefore

$$\frac{1}{\mu}\frac{d^2\widehat{xz}}{dx\,dz} = \frac{d^2}{dz^2}\left(\frac{du}{dx}\right) + \frac{d^2}{dx^2}\left(\frac{dw}{dz}\right).$$

Or

$$\frac{2}{\mu}\frac{d^2\widehat{xz}}{dx\,dz} = \left(\frac{d^2}{dz^2} + \frac{d^2}{dx^2}\right)\frac{\widehat{xx} + \widehat{zz}}{\lambda + 2\mu} + \left(\frac{d^2}{dz^2} - \frac{d^2}{dx^2}\right)\frac{\widehat{xx} - \widehat{zz}}{2\mu},$$

which gives us

$$\left(\frac{1}{\lambda + 2\mu} + \frac{1}{2\mu}\right)\left(\frac{d^4}{dx^4} + 2\frac{d^4}{dx^2 dz^2} + \frac{d^4}{dz^4}\right)V = 0.$$

Or finally the simple equation for V,

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dz^2}\right)^2 V = 0 \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(viii.)}.$$

(vi.) and (viii.) would be the general equations for the stresses in an "isotropic and homogeneous" dam; the arbitrary constants introduced into the solution must be found from the consideration that (*a*) the normal and shearing stresses on the horizontal top and curved flank are both zero, (*b*) the normal stress on the battered front is equal to the water pressure, and the shear is zero, and (*c*) either the stresses or the shifts must be supposed given over the base. It will be found at once that (i.) to (iii.) are not *absolutely* true even for a high vertical wall subject to fluid pressure on the front; there are modifying terms, similar to those first introduced into the flexure problem by DE SAINT VENANT for an isolated load,* and by PEARSON for a continuous load,† but these terms are of secondary importance. The shear is fairly closely given by (iv.). If we assume either (i.) to (iii.) or (iv.) alone to hold, it will be found that the relations (iv.) or (i.) to (iii.) are not true at the sloping front and flank. In fact, if we assume (i.) to (iii.), it is quite easy to determine the shear from the general conditions of equilibrium. For constant batter of both front and flank, the distribution of shear is still a parabola, but its axis no longer passes through the centroid of the cross-section. We have not assumed this distribution of shear, because there is just as much want of exactness in taking the tensile stresses as given by (i.) to (iii.) as in assuming the shear to be given by (iv.). It should not be at all an insuperable mathematical task to solve (vi.) and (viii.) for some simple forms of dam, front and flank.‡ If this were done, we should know much better the exact degree of approximation involved in the ordinary treatment of dams. The practical difficulty lies, as in all masonry structures, in the assumptions to be made as to the stresses on the base.

(3.) As far as the present investigation is concerned, we are not enquiring into the validity of the usual treatment; it is obviously faulty. But we are trying to indicate that, supposing it to be correct, its present *partial* application—to horizontal cross-sections only—involves the serious, and, we believe, often dangerous neglect of large tensions across the vertical sections.

To justify this statement, two model dams were prepared made of a fairly heavy wood, cut to a profile which we knew had been used in an actual case of construction. In the one case the dam was divided up into a series of horizontal strata, in the second case into a series of vertical strata. The first model dam appears to the right, the second to the left of our first plate. In the case of the dam stratified horizontally, the pull representing the water pressure was communicated by a cord to a stiff lath, which bore on the ends of the horizontal strata through two longitudinal strips of indiarubber tubing; the attachment of the cord was at one-third up the lath, and the cord was adjusted to be perpendicular to the battered front of the dam. In this way the resultant force was applied at the "centre of pressure," and was distributed much in the manner of water pressure. The cord, being carried over a vertical pulley, terminated in a bucket into which shot could be gradually dropped; this represented the resultant water pressure on the front. In the case of the dam stratified vertically, much the same arrangement was adopted, except that the pull of the cord was now applied directly to the first vertical stratum, which included the battered front.

---

* TODHUNTER and PEARSON, 'History of Elasticity,' vol. 2, pp. 53 *et seq.*

† 'Quarterly Journal of Mathematics,' vol. 24, pp. 63–110. See also PEARSON and FILON, *ibid.*, vol. 31, pp. 66–109.

‡ See the Appendix for a criticism of such a solution by M. LÉVY.

The coefficient of friction of the wood strips on each other varied somewhat; they had been left almost rough-sawn to increase its value. The angle of friction varied from stratum to stratum from about 25 degrees to 30 degrees; this is less than the 30 degrees to 36 degrees usually given for masonry on masonry. We could bring this well up to the masonry value by simply scattering silver sand between the strata, but this was not found possible in the case of the vertically stratified dam. We attempted pasting sand paper on the faces of the vertical strata, but this made the frictional strength far too great, the angle being then as much as 50 degrees. We obtained a shearing strength more nearly corresponding to masonry by pasting tissue paper round the battered fronts and curved flanks of the dams. Taking the actual dam on which these are modelled, the resultant water pressure on a strip of the front a foot broad is 312,500 lbs., and the weight of the corresponding portion of the dam is 505,000 lbs. The weights of the two model dams, although of the same dimensions, were not quite alike, the pieces of wood out of which they were cut being probably not of equal density. The horizontally stratified dam weighed 12·40 lbs., and the vertically stratified dam 13·85 lbs. This gives for the "water pressure" pulls on the above basis 7·70 lbs. and 8·57 lbs. respectively. Now we shall see in the sequel that, as far as theory goes, the line of resistance is within the middle third, but that the resultant force on the horizontal sections makes an angle of more than 30 degrees in some cases with the normal to the section. Hence we could not expect our model dams to reach the above water pressure values. As a matter of fact, trusting merely to the friction of wood on wood, the horizontally stratified one slid on its base at 5·70 lbs. pressure, and the vertically stratified one opened up at third section from the tail, and then the whole thing sheared at only 3·00 lbs. pressure. When the dams were strengthened for shearing resistance by tissue paper, as referred to above, the respective "water pressure" pulls before collapse were 6·50 lbs. and 4·20 lbs. The horizontal strata began to slide over each other at 6·50 lbs., and actual rupture of the paper took place at 8·25 lbs., not far from the 8·57 lbs. which the dam ought to carry. Various other experiments were tried, but the main features were the same, the horizontally stratified dam either sheared at its base or just above the tail between the third and fourth strata; in either case the giving at one or other of these sections was the signal for an approaching general collapse, as illustrated in the lower part of our plate. The vertically stratified dam opened up by tension very close to the tail, and then sheared over. It had to be watched very closely to follow the sequence of events, as the collapse was far more immediate and sudden than in the case of the horizontally stratified dam.* The nature of the rupture could be better exhibited by pasting the last two or three blocks together by tissue paper, and then the dam invariably opened up by tension at the next section, and showed this opening up in a marked manner.

---

* In the representation of the collapsing vertical dam, the dam was actually caught by the hands in the act of collapse and held till the parts were pasted together for the purpose of photographing.

The general results of these experiments were conclusive and obvious. They are as follows :—

(*a*.) The current idea that the critical sections of a dam are the horizontal sections is entirely erroneous. A dam collapses first by the tension on the vertical sections of the tail.

(*b*.) The shearing of the vertical sections over each other follows immediately on this opening up by tension.

(*c*.) It is probable that the shear on the horizontal sections is also a far more important matter than is usually supposed.

It follows from this that getting the line of resistance into the middle third of the horizontal sections, upon which all energy seems at present to be concentrated, is by no means the hardest and stiffest part of dam design. We should be much surprised if, with all the labour spent on this point, the bulk of existing dam constructions are not for masonry under very considerable tension, *i.e.*, a tension across the vertical sections, which has been hitherto disregarded.

We propose therefore to lay it down as a rule for the construction of future dams that the stability of the dam from the standpoint of the vertical sections must be considered in the *first* place. If this be satisfactory, we believe that the horizontal sections will be found to be stable, but of course the latter must be independently investigated.

(4.) We shall illustrate this by the combined graphical and analytical treatment of the dam from which our model was taken.

Starting with the portion of the dam between two transverse sections a foot apart, the dimensions are as in the accompanying figure, the height being 100 feet, and the water on the front extending to the top. The weight of a cubic foot of the masonry is taken to be 2·25 times that of a cubic foot of water. We neglected the dotted portion of the tail, although it might have been better to include it, but it can make no sensible difference to the main results ; one of its chief influences will be to cause the distribution of shear over the base to be really more like the value assumed in (iv.) than it would otherwise be.

Turning to the algebraical treatment of the matter, let us assume what is exceedingly nearly the truth for a considerable distance from the tail, *i.e.*, that the flank is given by a straight line. Let the total vertical force on the basal horizontal section be Q_0, and the total horizontal force on the section be P_0, and let them act at a definite load-point determined by the intersection of the line of resistance for full reservoir with the basal section. The pressures on the base are given by a straight line cutting the base, by (i.), at a distance d from the centroid $= \frac{1}{3}b^2/c$, where $2b$ is the length of the base, and c is the distance of the load-point from the centroid. The

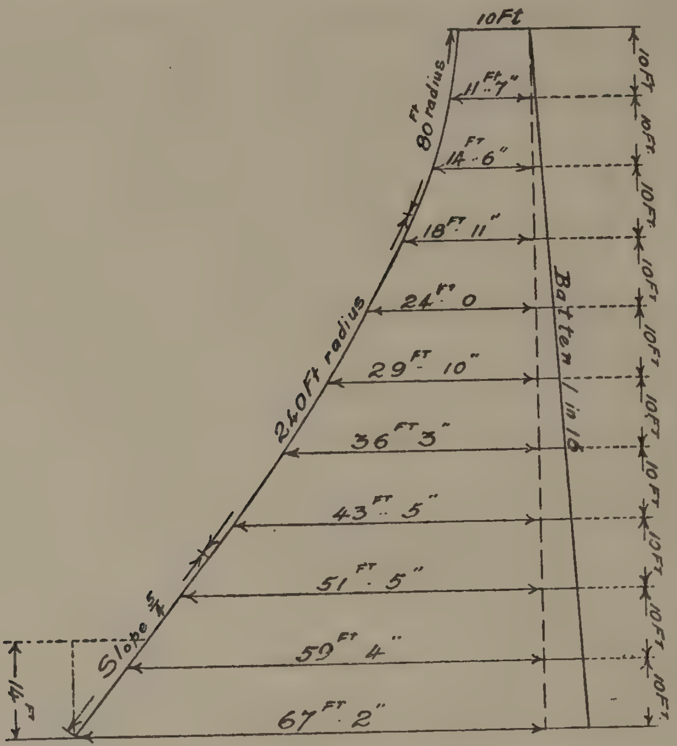

Fig. I. Section with Dimensions of an actual Dam.

equation to the line giving the distribution of pressures over the base is then, if y be the vertical height-giving pressure and the origin the centroid,

$$y = \frac{Q_0}{2b}\left(1 - \frac{x}{d}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{ix.}).$$

If Q_0 and P_0 be measured as cubic feet of masonry, this will give the tension y as a height of masonry.

If we transfer the origin to the tip of the tail, the equation is

$$y = \frac{Q_0}{2b}\,\frac{d + b - x}{d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{x.}).$$

The equation to the flank of the dam, so far as it is linear, may be taken as

$$y = mx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xi.}).$$

Now, if we subtract these two lines from each other, we shall have their difference represented by the line

$$y = \frac{Q_0(d + b)}{2bd} - x\left(m + \frac{Q_0}{2bd}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xii.}),$$

or the area inclosed between this line and the base represents the total upward force in cubic feet of masonry on any length x of the tail measured from the base.

Thus the total upward force F_1 on the length x of tail is

$$F_1 = \frac{Q_0 (d + b) x}{2bd} - \frac{x^2}{2}\left(m + \frac{Q_0}{2bd}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xiii.)}.$$

Its vertical line of action is given by

$$x = x \frac{\frac{1}{2}Q_0 (d + b) - \frac{1}{3}x (2bdm + Q_0)}{Q_0 (d + b) - \frac{1}{2}x (2bdm + Q_0)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xiv.)}.$$

Next let us turn to the distribution of shear. This is regulated by (iv.). Therefore, in our case,

$$S = \tfrac{3}{2}\frac{P_0}{2b}\left(1 - \frac{(b - x)^2}{b^2}\right)$$

$$= \tfrac{3}{2}\frac{P_0}{2b}\left(\frac{2bx - x^2}{b^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xv.)}.$$

Thus the total shear F_2 over a length x of base from the tail is given by

$$F_2 = \tfrac{3}{2}\frac{P_0}{2b}\frac{bx^2 - \frac{1}{3}x^3}{b^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xvi.)}.$$

We have now the total "external" forces on the portion of tail on the base-length x.

These must be kept in equilibrium by the resultant of the stresses over the vertical section of the dam at distance x from the tail. Hence we find :

Total shearing stress on vertical section at distance x

$$= F_1 = \frac{Q_0 (d + b) x - \frac{x^2}{2}(2bdm + Q_0)}{2bd} \quad . \quad . \quad . \quad . \quad . \quad \text{(xvii.)}.$$

Total thrust on vertical section at distance x

$$= F_2 = \tfrac{3}{2}\frac{P_0}{2b}\frac{bx^2 - \frac{1}{3}x^3}{b^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xviii.)}.$$

Further, the point at which the resultant of F_1 and F_2 cuts the vertical section of the dam at x from the tail is given by

$$\bar{y}/(x - \bar{x}) = F_1/F_2,$$

or the load-point of the x-section is determined by

$$\bar{y} = \frac{b^2}{3dP_0}\frac{Q_0 (d + b) - \frac{1}{3}x (2bdm + Q_0)}{b - \frac{1}{3}x} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xix.)}.$$

But this relation between $\bar{y}$ and x is clearly the equation to the line of resistance of the vertical sections as long as the flank is given by a straight line. The line of resistance is accordingly a hyperbola. The vertical asymptote is

$$x = 3b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xx.)},$$

and the horizontal asymptote

$$y = \frac{2bdm + Q_0}{P_0} \frac{b^2}{3d} = \frac{2bdm + Q_0}{P_0} c \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxi.)}.$$

The curve will be above or below its asymptote at the start on the axis of y, according as

$$\frac{Q_0(d + b)}{P_0 b} \frac{b^2}{3d} > \text{or} < \frac{2bdm + Q_0}{P_0} \frac{b^2}{3d},$$

or, as

$$Q_0 > \text{or} < 2b^2 m.$$

We must now consider the stresses on the vertical sections.

First.—The maximum shearing stress is certainly greater than, and probably not very different from, 1·5 into, the mean shearing stress, or $= \frac{3}{2}F_1/\zeta$, if ζ be the height of a section. But $\zeta = mx$, hence

Maximum shear on vertical section at distance $x = \frac{3}{2} \dfrac{Q_0(d+b) - \frac{1}{2}x(2bdm + Q_0)}{2bdm}$. (xxii.).

This is a straight line, such that Tb (Plate III.), the shear at the tail, is $\frac{3}{2} Q_0 (d + b)/2bdm$ in height of masonry. Of course, this shows the necessity for thickening the tip of the tail, as is actually done in the dam under consideration.

Secondly.—The maximum tensile stress is given by

$$T_{\text{max.}} = -\frac{F_2}{\zeta} + \frac{\frac{1}{2}\zeta M}{\frac{1}{12}\zeta^3}$$

$$= \frac{F_2}{\zeta}\left(-1 + \frac{\bar{y} - \frac{1}{2}\zeta}{\frac{1}{6}\zeta}\right) = \frac{6F_2}{\zeta^2}\left(\bar{y} - \tfrac{2}{3}\zeta\right).$$

Hence substituting for F_2, for $\bar{y}$ and mx for ζ we have,

$T_{\text{max.}}$, on vertical section at distance x from tail

$$= \frac{3}{2} \frac{1}{bdm^2}\left\{Q_0(d + b) - \tfrac{1}{3}x(2bdm + Q_0)\right\} - \frac{3P_0}{b^3 m} x\left(b - \tfrac{1}{3}x\right) \quad . \quad \text{(xxiii.)}.$$

This is clearly a parabola with vertical axis.

If T_0 be the value of the maximum tensile stress at the tail,

$$T_0 = \frac{3Q_0(d + b)}{2bdm^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxiv.)}.$$

Further, if x_0 mark the distance from the tail at which the line (xii.) cuts the base, and y_0 the height of this line at the tail,

$$x_0 = \frac{Q_0(d+b)}{2bdm + Q_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxv.)},$$

and

$$y_0 = \frac{Q_0(d+b)}{2bd} = \tfrac{1}{3}m^2 T_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxvi.)},$$

we have,

$$T_{\text{max.}} = \frac{3y_0}{m^2}\left\{1 - \tfrac{1}{3}\frac{x}{x_0} - \epsilon\frac{x}{b}\left(1 - \tfrac{1}{3}\frac{x}{b}\right)\right\} \quad . \quad . \quad . \quad . \quad . \quad \text{(xxvii.)},$$

where

$$\epsilon = P_0 m/(by_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxviii.)},$$

or

$$T_{\text{max.}} = T_0\left\{1 - \tfrac{1}{3}\frac{x}{x_0} - \epsilon\frac{x}{b}\left(1 - \tfrac{1}{3}\frac{x}{b}\right)\right\} \quad . \quad . \quad . \quad . \quad . \quad \text{(xxix.)}.$$

The vertex of this parabola is at

$$x = \tfrac{3}{2}b + \frac{1}{2\epsilon}\frac{b^2}{x_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxx.)},$$

and the value T_1 of $T_{\text{max.}}$ at this point is

$$T_1 = T_0\left\{1 - \frac{1}{12\epsilon}\left(3\epsilon + \frac{b}{x_0}\right)^2\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxxi.)}.$$

Thus the values of the ordinates of this parabola at $x = 0$ and at the vertex being known, it can be easily drawn.

The construction of the line of maximum shearing stress and the parabola of maximum tensile stress for the vertical sections of the dam is by far the easiest procedure, so long as the flank is linear. After the flank ceases to be linear about the point B of Plate III., we must fall back upon the purely graphical processes, to be now described.

(5.) We proceed to indicate the graphical stages leading up to the constants required in the above investigation. The section of the dam was first divided horizontally into 10 blocks of equal height, and the central line of the dam and the middle thirds put in. The centroids of each of these trapezia were then determined, and the centroid verticals put in. This is all done on the extreme right of Plate II. The weights of each of these masonry trapezia—depth 10 feet—were then plotted down the vertical 0 . . . 10, and with pole O the link polygon on the lower right-hand corner of Plate II. obtained.

The intersections of the successive links with the first link to the right of this link polygon give by vertical lines the lines of action of the weights of all the masonry above the horizontals 1, 2, 3 . . . 10 respectively. Producing these lines to meet the corresponding horizontals, we obtain the line of resistance of the reservoir when empty, and see that, as far as the condition of falling within the middle third is concerned, the empty reservoir is stable. Next, the centres of pressure for all the water down to the horizontals 1, 2, 3 . . . 10 respectively were determined, and the values of the water pressures—total area × pressure at centroid—were determined and are marked as W_1, W_2 . . . W_{10}. These water pressures are plotted in lines perpendicular to the front of the dam, through the points 1, 2, 3 . . . 10 of the vector polygon already referred to, and the resultant force on each horizontal section is thus determined. The lines of action of each water pressure on the front of the dam are then produced to meet the corresponding weight verticals, and through the points thus determined lines are drawn parallel to the corresponding resultant forces until they meet the corresponding horizontals. The points so obtained, when joined up, give the line of resistance for the dam when the reservoir is full. We see again that this line lies in the middle third, or the dam satisfies this current condition of stability. All this work is perfectly familiar, and is part of the usual graphical treatment. The resultant water pressure and the resultant weight, if combined and resolved horizontally and vertically, give the total shear on the base, and the total thrust; these are found to be 312,000 lbs. and 523,000 lbs., the resultant water pressure being 312,500 lbs., and the total weight of the dam 505,000 lbs. In cubic feet of masonry we have them as:

$$P_0 = 2228\cdot6 \text{ cubic feet and } Q_0 = 3735\cdot7 \text{ cubic feet.}$$

Next, the total shear and total thrust on each individual horizontal section was taken by resolving horizontally and vertically each corresponding resultant force. The maximum shear on any horizontal section was then taken as $\frac{3}{2}$ the mean shear over its area. These maximum shearing stresses are plotted to the right of Plate III., the maximum shearing stress being on the base and less than 3 tons per square foot. A knowledge of the load-point of each horizontal section (given by the intersection of the section with the line of resistance) and of the thrust will enable us to find from (ii.) the maximum compressive stress. These stresses are plotted also on the right of Plate III. for each horizontal section. The maximum compressive stress is at the base, and is only a little over 5 tons per square foot.

Lastly, if we consider the usual frictional test with an angle of friction of 30 degrees, the resultant on any horizontal cross-section ought not to make an angle of less than 60 degrees with the cross-section. The last four sections will be found to make angles slightly less than 60 degrees, and therefore, if 30 degrees be taken as the limit, the material would be holding together by the shearing strength, which is probably ample.

Thus the whole dam, if treated by the usual processes, would be declared satisfactory and stable. Indeed, it is far more so than a good many profiles we have come across; it is considerably more stable than RANKINE's standard section. We now pass, however, to consider the thus far neglected vertical sections.

In the left-hand bottom corner of Plate II. the parabola of shear on the base (xv.) is drawn; this is integrated by the usual graphical process and the sum curve of shear found, giving the total shear up to any vertical cross-section. This is the graphical equivalent of (xvi.). Next, the curve of pressures on the base, deduced from (i.) and (ii.), with its neutral axis is drawn. This is done on the upper left-hand diagram of Plate II. If this curve be plotted in height of masonry, the area between it and the flank of the dam will represent the total upward force on the tail of the dam up to any section. After the "Curve of Pressures on Base" and the flank of the dam cross, the forces will begin to be downwards. This area was divided up vertically into strips, and the area of these strips plotted first down, and, when they became negative, up, a vector line. This is the left-hand vector figure 0, 1, 2, 3 . . . 24 of the middle of Plate II. A pole O′ being taken, a link figure was drawn for the weights 1 to 19. To keep the figure clear, a second pole O″ was taken and a separate link figure for the negative areas 19 to 24. This appears somewhat above the first link figure. The construction gave, by the intersection of the successive links with the horizontal, the position of the resultant upward forces on the dam up to any vertical section at distance x from the tail, *i.e.*, we obtained the $\bar{x}$ values of (xiv.). The total vertical force was given by the corresponding length on the vector figure. The total corresponding shears taken from the sum curve of the shear parabola* were plotted horizontally as 01, 02, and, being combined with the total upward forces, gave in magnitude and direction the resultant force on a vertical section. Parallels to these resultant forces, through the points where the corresponding resultant upward forces met the base of the dam, determined by their intersections with the corresponding vertical sections the load-points, and so the line of resistance VWZ. This process is indicated for the 14th force, giving the point W on the drawing. It was found that this process was not capable of being applied with very great accuracy in the *immediate* neighbourhood of the tail of the dam; hence as long as the flank of the dam was linear the hyperbola given by (xix.) was constructed, and this was found to run into the graphically-determined system of load-points quite closely before the end of the linear flank was reached.

The line of resistance VWZ being known, and the vector figure (left-hand vector polygon on the upper middle part of Plate II.) giving by resolution the horizontal or total thrust and the vertical or total shearing components on the vertical sections, it became easy to determine the maximum shearing and maximum tensile stresses on the vertical sections. Reduced from cubic feet of masonry to tons per square foot, these stresses are plotted in the "Curve of Maximum Shearing Stresses" and "Curve of Maximum Tensile Stresses (I)" of Plate III. Tu is the base of these curves, and we see that the straight line and parabola of the linear part of the flank run into the graphical determinations. The effect of the curved flank after B is to throw the curve of maximum shear below the straight line (dotted) after B and the curve of maximum tensile stress below the parabola (dotted) after B; in other words, it reduces somewhat the dangerous stresses.

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* The ordinates of the sum curve of shear give total shears to *half* scale of upward forces, and therefore needed doubling before plotting horizontally.

Now these curves give most important results. We see :—

(i.) That the line of resistance for the vertical sections lies outside the middle third bounded by AJ′Y right away up to the point M, or the vertical section MI. In other words, rather more than half the vertical sections are subjected to tension.

(ii.) That the tensile stresses in the tail are for masonry very serious, amounting to nearly 10 tons per square foot at the extreme tip, and to 6 tons per square foot after we have passed the vertical section, where the strengthening of the tail has ceased.

(iii.) That the maximum shearing stresses amount to 6 tons per square foot at the tip of the tail and 5 tons per square foot after we have passed the vertical section, where the strengthening of the tail has ceased. We lay no stress on the actual values of these "maximum" shears on the vertical sections, as they are obtained from the mean shears by using the round multiplier 1·5. This round number is assumed because the maximum is certainly greater than the mean shear. The actual distribution of shear on the vertical sections has not been discussed. It could of course be found from that on the horizontal sections, if the latter were really known with sufficient accuracy, by the equality of the shears in two planes at right angles. We content ourselves, however, by showing that the *mean* shears on the vertical sections appear to be higher than those on the horizontal sections, and thus indicate that the parabolic distribution, applied to sections some way above the base, probably *under-estimates* the maximum shearing stresses in the dam.

In other words: *Whether we judge by the line of resistance lying outside the middle third, or by the existence of serious tensile stresses, or by the magnitude of the mean shearing stresses, the vertical sections are critical for the stability in a far higher degree than the horizontal sections.*

In a well-designed dam, all the conditions for stability of the horizontal sections may have been satisfied, yet if the very same conditions be applied to the vertical sections not one of them will be found to be satisfied. It seems accordingly very unsatisfactory that the current tests for stability should, if they are legitimate, be applied to the horizontal instead of to the far more critical vertical sections. In the case of the latter they fail completely; and if high tension and shear are to be allowed in the vertical sections, then it is absurd to exclude them in the case of the horizontal sections. We maintain that the current treatment of dams is fallacious, for it screens entirely the real source of weakness, namely, in the first place the tension, and in the second place the substantial shear in the vertical sections, and this at distances from the tail far beyond the usual tail strengthening range.

Nor do these theoretical results stand unverified by experiment; they are absolutely in accord with our experience on the model dams. These dams collapsed precisely as we might expect from the above investigation, *i.e.*, the dam with vertical sections gave long before the dam with horizontal sections. The former collapsed by opening up of the joints by tension towards the tail, followed almost immediately by a shear of the whole structure. In the case of the horizontally stratified dam, the collapse, which occurred much later, was by shear of the base, followed almost simultaneously by a shear of one or more of the horizontal sections.

(6.) We place here the actual numerical constants for the dam under investigation.

P_0 = total shear on base = 312,000 lbs. = 2228·6 cubic ft. of masonry.
Q_0 = total thrust on base = 523,000 lbs. = 3735·7 cubic ft. of masonry.
$b = \frac{1}{2}$ breadth of base = 36·68 ft.
d = distance of neutral axis of base from centroid = 62·24 ft.
c = distance of load point of base from centroid = 7·21 ft.
m = slope of linear portion of flank to base = 5/4.
$2bdm + Q_0$ = 9443·1 cubic ft. of masonry.
$y_0 = 80{\cdot}93$, $x_0 = 39{\cdot}13$, these give the line of equivalent masonry for upward forces on tail. See (xii.), (xxv.), and (xxvi.).
$\epsilon = {\cdot}938$. See (xxviii.).

We are now able to construct our curves.

The equation to the hyperbolic line of resistance (xix.) is

$$y(1 - {\cdot}00909x) = 32{\cdot}59 - 30{\cdot}55 \times {\cdot}00909x,$$

the units being feet.

Accordingly, the line of resistance at the tip of the tail is 32·59 feet above the base of the dam, and its asymptote falls below it at 30·55 feet above the base. The hyperbola is a very flat curve, and in fact does not deviate very widely from a horizontal line through the centre of pressure. The equation to the line of maximum shear (xxii.) is

$$\text{Maximum shear} = 97{\cdot}116 - 1{\cdot}2409x$$

in height of masonry, the units being feet. This height must be multiplied by 140 lbs. or $\frac{1}{16}$ ton to obtain lbs. per square foot of stress. Lastly, we have for the parabola of maximum tensile stress (xxix.):

Distance of vertical axis from tail = 93·35 ft.
Tension T_0 at tip of tail = 155·39 ft. or 9·71 tons.
Tension T_1 at vertex = − 38·85 ft. or − 2·43 tons.

These data suffice to construct the parabola.

(7.) We now turn finally to another point. In how far does the distribution (iv.) assumed for the shear on the basis affect the main features of the result? We have already indicated that the distribution of the stresses corresponding to a neutral axis and load-point gives a parabolic distribution of shear, but one that differs from (iv.), the shear not absolutely vanishing at the terminals. But this distribution of normal stresses is no more true than the distribution of shearing stresses, and there is no more reason for assuming the one to be absolutely true and correcting the other than *vice versâ*. Both need corrective terms like those applied by DE SAINT VENANT in the flexure of cantilevers. Until a number of solutions have been found for (vi.) to (vii.) we cannot say how far our normal stress distribution, or how far our shearing stress distribution is in error. But we may quite safely assert that (iv.) must be far closer to the actual truth than, for example, the assumption that the shearing distribution is *uniform* over the base. A little consideration shows at once that the parabolic distribution must far more nearly represent the physics of the case. It seemed worth while, however, to take the extreme case of uniform shear over the base and see how far this would modify the conclusions reached on an assumption which may be justly said to give the "true" shear as compared with this extreme case. Let us consider the analysis of this case for the portion of the flank which is linear.

Equations (ix.) to (xiv.) still remain true. Equation (xv.) is replaced by

$$S = \frac{P_0}{2b} = S_0, \text{ say.}$$

This leads to the expression

$$F'_2 = S_0 x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxxii.})$$

for the total thrust on a vertical section at x. For the total shear on a vertical section at x, (xvii.) remains in force and the curve of maximum shear of (xxii.), or as drawn in Plate III. is unchanged. Hence this new hypothesis does not alter the large shearing stresses on the vertical sections already noticed. Instead of (xix.) we have, however,

$$\bar{y}'(x - \bar{x}) = F_1/F'_2, \quad \text{or,}$$

after some reductions this leads to

$$\bar{y}' = \frac{m}{\epsilon}\left(x - \frac{1}{3}\frac{x^2}{x_0}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxxiii.}).$$

Thus we see that the line of resistance is a parabola passing through the tip of the tail, and having its vertex at a distance $\frac{3}{2}x_0$ from the tip, with a height (qQ of Plate III.) $\frac{3}{4}mx_0/\epsilon$.

For the actual dam these are 58·695 feet and 39·11 feet. We see that the line of resistance lies outside the middle third for more than half the dam. The dotted curve NZ gives the line of resistance determined graphically after it leaves the parabola, *i.e.*, when the flank becomes curved. The deviation is not very great, and we see that, as before, about half the dam has to carry tension across its vertical sections. We now take the maximum tensile stress across the vertical sections; we have, as before,

$$T_{\text{max.}} = \frac{6F'_2}{\zeta^2}\left(\bar{y}' - \frac{2}{3}\zeta\right).$$

Or, using the notation of (xxv.) to (xxviii.), we find, after some reductions,

$$T_{\max.} = T_0 \left\{1 - \tfrac{2}{3}\epsilon - \tfrac{1}{3}\frac{x}{x_0}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxxiv.)}.$$

This is a straight line. It may be drawn by noticing that it gives a tension of $T_0(1 - \frac{2}{3}\epsilon)$ at the tip of the tail, and that the tension vanishes, *i.e.*, the line of resistance for a straight flank would come into the middle third, at $x = (3 - 2\epsilon)x_0$. For the particular dam under consideration these values are 3·64 tons and 43·98 feet. This line of tensile stresses is drawn on Plate III. and is completed as a curve JL by graphical calculations after the flank ceases to be linear at J. It will be seen that even in this case there would be a tension of between 3 and 4 tons at the tip of the tail and of 2 to 3 tons after we have passed the portion, about 11 feet, for which the tail is thickened up.

Now we have here taken a distribution which must be far further from the actual distribution of shear than the first hypothesis. Yet we still find :—

(i.) That the line of resistance falls well outside the middle third for about half the dam.

(ii.) That there exist considerable tensions, 3 to 4 tons per square foot, in the masonry.

(iii.) That the average shearing stresses on the vertical sections are greater than on the horizontal sections. As a result of this extreme case we think we may safely say that the real distribution of shear over the base, whatever it may be, must lead us to a line of resistance lying well outside the middle third, and to tensions amounting to something between 5 and 10 tons per square foot.

(8.) *General Conclusions.*

(1.) The current theory of the stability of dams is both theoretically and experimentally erroneous, because :—

- (*a.*) Theory shows that the vertical and not the horizontal sections are the critical sections.
- (*b.*) Experiment shows that a dam first gives by tension of the vertical sections near the tail.

(2.) An accepted form of profile is shown to be stable as far as the horizontal sections are concerned, but unstable by applying the same conditions of stability to the vertical sections.

(3.) The distribution of shear over the basis must be more nearly parabolic than uniform, but as no reversal of our statements follows when we pass from the former to the latter extreme hypothesis, it is not unreasonable to assume the former distribution will describe fairly closely the facts until we have greater knowledge.

(4.) In future we hold that in the first place masonry dams must be investigated for the stability of their vertical sections. If this be done we feel pretty certain that most existing dams will be found to fail, if we accept the criteria of stability usually adopted for their horizontal sections. This failure can be met in two ways:—

(*a.*) By a modification of the customary profile. We think it probable that a profile like that of the Vyrnwy dam would give better results than more usual forms.

(*b.*) By a frank acceptance that masonry, if carefully built, may be trusted to stand a definite amount of tensile stress. It is perfectly idle to assert that it is absolutely necessary that the line of resistance shall lie in the middle third for a horizontal treatment, when it lies well outside the middle third for at least half the dam for a vertical treatment.

We hope to return later to the question of the best dam profile based on a consideration of the stability of the vertical sections. Meanwhile, without being unduly alarmist, we should like to draw the attention of practical engineers to the point that the masonry dams they have designed to be, according to a defective theory, free from tensile stress are probably subjected not only to tensile, but to shearing stresses far exceeding anything they have in the least imagined. We may even speculate whether this truth may not be to some extent the source of weakness or even of collapse in certain dams which have existed in the past.

APPENDIX.

On M. LÉVY'S *Solutions of the Dam Problem.*

When the above graphical treatment of the dam problem was being developed, we were unaware that M. MAURICE LÉVY had published anything further on the subject than appears in his 'La Statique graphique,' cited on p. 6. In consulting, however, the 'Annales des ponts et chaussées' for 1903, upon another matter, we came across a technical paper developing an analytical solution due to M. LÉVY. Looking up the references provided we found the following papers by M. LÉVY:—

(*a.*) "Quelques considérations sur la construction des grands barrages," 'Comptes rendus de l'Académie des Sciences,' tome 121, pp. 288–300, Paris, 1895.

This memoir assumes the normal stress on a horizontal section to be linear, and deduces the result that the shear on the horizontal sections will follow a parabolic distribution not vanishing at the front and flank. This is the result re-discovered by us and cited on p. 9. There is, however, a point to be regarded here. Suppose the dam to be on a horizontal substratum. Then the distribution of shear on this substratum is really the most important question in the shearing stress. On the substratum just outside the flank and just outside the front the shear is undoubtedly zero. Hence, unless we suppose discontinuity at flank and front, the shear cannot rise suddenly to a finite value. In most cases the flank rises perpendicularly to the substratum for some feet, and quite possibly this may occur at the front if the contour starts with a step. If this be so, the parabola of shear must vanish at both flank and front, and take the form indicated in our equation (iv.). We have already stated that we find no more grounds for adopting a linear distribution of normal stress than a central parabolic distribution of shear; both are actually modified by quite unknown corrective terms. Hence we have retained our central parabolic distribution of shear, fully realising that at flank and front it must to some extent be modified, as we also believe the linear distribution of normal stresses certainly will be.

(*b.*) *Note de* M. MAURICE LÉVY: "Sur la légitimité de la règle dite du trapèze dans l'étude de la résistance des barrages en maçonnerie," 'Comptes Rendus,' tome 126, pp. 1235–1240, Paris, 1898.

In this memoir LÉVY mentions the "catastrophe de Bouzey," and insists on the importance of further consideration of the stresses in dams. He gives what he apparently considers a close approximation in the case of a dam of rectangular cross-section with water to the top. Thus, if the dam be of thickness τ and the origin be where the top meets the front, he finds for solutions of our equations (v.), p. 7, the unit being 1 ton of water per square metre and γ the specific gravity of the masonry:

$$\widehat{zz} = -\gamma z - \frac{z^3}{\tau^2}\left(\frac{2x}{\tau} - 1\right) - z\left(-\frac{4x^3}{\tau^3} + 6\frac{x^2}{\tau^2} - \tfrac{12}{5}\frac{x}{\tau} + \tfrac{1}{5}\right),$$

$$\widehat{zx} = -\frac{x}{\tau}\left(1 - \frac{x}{\tau}\right)\left\{\frac{3z^2}{\tau} - \frac{\tau}{5} + x\left(1 - \frac{x}{\tau}\right)\right\},$$

$$\widehat{xx} = \left(1 - \frac{x}{\tau}\right)^2\left(1 + 2\frac{x}{\tau}\right)z.$$

This fulfils perfectly the conditions at front and flank. At the top where $z = 0$, we have not $\widehat{zx} = 0$, but

$$\widehat{zx} = \frac{x}{\tau}\left(1 - \frac{x}{\tau}\right)\left\{\frac{\tau}{5} - x\left(1 - \frac{x}{\tau}\right)\right\}.$$

This forms a system of shearing stresses in equilibrium which vanishes at four points and has maxima of $-\tau/100$ and $-\tau/80$, so that LÉVY assumes that the unbalanced stresses on the top are negligible. LÉVY appears to consider that with this exception he has obtained an exact solution of the dam problem, but this is very far from the case. Our criticism applies not only to this but to later solutions of LÉVY. It is of the following nature. The full solution of an elastic problem in uniplanar stress must involve the discovery of the body stresses due to the superficial loading when the latter is given *at every point of the closed contour.* Now LÉVY has found a solution which involves a certain loading on a portion of the contour, namely, the base of the dam, which distribution he has not examined or justified in any manner. If this distribution be not the actual distribution—and he gives no evidence that it is—his solution falls entirely to the ground. Consider any block of elastic material, say the shape of the dam in our figure on p. 12, and suppose any system of stresses applied to the base *only* (for example, an upward pressure of 2 p lbs. per foot run on the middle third and a downward pull of p lbs. on the outer thirds), which forms an equilibrated system, then the elastic block will be in equilibrium under this system with no other body or surface forces acting. Accordingly we may add to LÉVY'S solution any other one arising from any such equilibrated system of stresses on the base. Now, were we dealing with a rod, of which the linear dimensions of the cross-section were small as compared to the length, any such equilibrated system of stresses on a terminal cross-section would produce small effect on the rod at a little distance from the terminal. But in the case of a dam of which the dimensions of the base are of the same order as the height, the internal stresses produced by an equilibrated system of forces applied to the base may be of a very pronounced character, especially at the tail of the dam, which is relatively near to the base. Thus we may add to LÉVY'S solution of the rectangular dam the internal stresses produced by any system of equilibrated stresses over the base of the dam, and the resulting solution will be equally valid, or equally invalid, with his; at any rate, this statement will be true until we have some experimental or theoretical supposition to make with regard to the nature of the stresses (or the shifts?) of the substratum under the dam. LÉVY'S solution would be an approximate solution for a "high wall," because in such a case the equilibrated system at the base would be of small importance at some distance from it. It is not a justifiable solution at all in the case of dams of very sensible base dimensions. He has, in fact, assumed the shearing stress and pressural stress over the base to be given by certain biquadratic forms. There does not appear to be any further justification for this than the desire to get a complete solution of some kind. There would clearly be a solution if we supposed the shearing stress and the pressural stress to be respectively parabolic and linear over the base and at the same time in equilibrium with the applied forces. *A priori* such forms are as legitimate as LÉVY'S biquadratic distributions, only the corresponding expressions for the internal stresses are not easy to find analytically as in LÉVY'S arbitrary case. We have here really an instance of the difficulty we have already referred to on pp. 1 and 9 of this memoir, which renders the application of the mathematical theory so difficult—we cannot predict the nature of the reactional stresses over the large areas of support in the case of masonry; this is a difficulty which is immensely intensified as we pass from arch to dam. We can only hope for a series of solutions of equations (vi.) and (vii.), so that we may judge how the stresses change with different hypotheses as to the base distribution of pressure. In this sense M. LÉVY'S solution is of value because it is suggestive, but how far it fails from being general is shown by our experimental dam, which opened by tension. Hence there may be a real element of danger in a solution which asserts no tension to exist in the structure.

La règle du trapèze, to which LÉVY refers in the title to his second paper, is one apparently familiar to French engineers, which, besides making the tensions on the horizontal sections *linear*, supposes the shears also to be linear. Assuming his solution given above to be the true solution, LÉVY shows that the maximum value of the stress obtained from it is less, and the minimum value greater than that given by the trapezium rule. Hence he argues that that rule errs on the side of safety. The error of this conclusion seems to us to lie in the assumption that the above is the *true* solution.

(*c*.) "Sur l'équilibre elastique d'un barrage en maçonnerie à section triangulaire," 'Comptes Rendus,' vol. 127, pp. 10–15, Paris, 1898.

Suppose the water to reach the apex of the dam and the front to make an angle α, the flank an angle with the vertical, then Lévy gives solutions satisfying the front and flank conditions (but these *only*) when α is zero and again when α has any value. For the first case the stresses are:

$$\widehat{zz} = -(\gamma - \cot^2\beta)\,z - (2\cot^3\beta - \gamma\cot\beta)\,x,$$

$$\widehat{zx} = -x\cot^2\beta,$$

$$\widehat{xx} = -z.$$

In the second case the stresses are still linear in x and z, but complicated functions of α and β. Lévy speaks of these as exact solutions and supposes them, being linear, to justify the trapezium rule. Clearly no part of the dam will be in tension. But the real facts of the case seem to us to be that Lévy has got his solution by supposing the stresses on the base to be linear, and such hypothesis has *à priori* no greater justification than one which assumes them to be quadratic or biquadratic. Lévy's solution allows of no tension in such a triangular-sectioned dam. Our model dam, which is not very far from triangular in form, does show such tension; in fact, a little consideration of such a simple case as that of $\gamma = 0$ shows that $\widehat{xx}$, being always negative, depends entirely on the arbitrary assumption made that the reactional stresses on the base are linear. Thus it will be seen that Lévy's solution for the dam of triangular cross-section is no more "exact," as far as actual dams are concerned, than his previous solution for the rectangular section. Under the circumstances, we did not consider it needful to modify our hypotheses or graphical constructions.

If it be asked: Why is our parabolic distribution of shear on the base any more justifiable than the linear? the answer must be threefold.

(i.) We have not confined our attention to a parabolic distribution, but considered also a linear uniform distribution as another extreme.

(ii.) Both lead to a sensible tension in the tail, which would fail if the solution of Lévy were correct, but such tension is confirmed by experiment.

(iii.) If the dam be built, as most dams are, so that the tip of the tail is vertical (see Fig. I.), then the linear solution of Lévy, if applied to the base, would give a maximum shear on the base *at the flank*, whereas actually this shear must be zero. Indeed, the dam near the base is now like a vertical wall with pressure on one side, and therefore the distribution of shear will now approach nearer to that given by Lévy for the dam of rectangular cross-section, *i.e.*, be more nearly parabolic in form.

Now suppose we assume the normal stresses on the horizontal sections to be linear, in the manner defined by (i.) to (iii.), then we readily deduce:

(*a.*) For the dam of triangular section, vertical front

$$\widehat{zz} = -(\gamma - \cot^2\beta)\,z - (2\cot^3\beta - \gamma\cot\beta)\,x.$$

This agrees absolutely with Lévy's solution.

(*b.*) For the dam of rectangular section:

$$\widehat{zz} = -\gamma z - \frac{z^3}{\tau^2}\left(\frac{2x}{\tau} - 1\right).$$

This agrees with the first two terms of Lévy's solution.

The ratio of Lévy's additional third term to our second term above is the ratio of $\frac{z^2}{\tau^2}$ to $2\left(\frac{x}{\tau} - \frac{1}{2}\right)^2 - \frac{3}{10}$, but both vanish when $x = \frac{1}{2}\tau$ or at the middle of the horizontal sections. At quarter sections the ratio of the terms is 100 to $17 \cdot 5 \times \tau^2/z^2$, and at flank and front 100 to $20 \times \tau^2/z^2$. Thus, if the rectangular section had the height double of its breadth, Lévy's term would introduce on the base of the dam something under a 5-per cent. modification of the normal stresses as deduced from (i.) to (iii.). This is certainly well within the limits of alteration due to any equilibrated system of stresses applied to the base. The more the dam diverges from the rectangle and approaches the triangle, the more closely (i.) to (iii.) would apply, supposing LÉVY'S solution were not a special, but the true solution. We conclude accordingly that for any form of dam section lying roughly between rectangle and triangle, we may apply (i.) to (iii.) or the theory of load-point and neutral axis to obtain the normal stresses on the horizontal sections near the base; and this will be approximately true for all horizontal sections, if the upper part of the dam be fairly triangular in shape. The divergence from this distribution will depend on the actual stress distribution over the base, which must be considerably influenced by the elasticity of the substratum and the manner in which the dam is gradually built up from this substratum. Roughly, we might say that the deviation will probably be less than 5 per cent. of the hypothetical values. This forms (as far as anything is yet possible on the basis of elastic theory) the justification of (i.) to (iii.).

We now turn to the question of shear. This is to be found from the fundamental equation in (v.), *e.g.*,

$$\widehat{xz} = -\int\left(\gamma + \frac{d\widehat{zz}}{dz}\right)dx.$$

This gives us:—

(*a.*) For the triangular-sectioned dam, remembering that the shear vanishes at the front,

$$\widehat{xz} = -x\cot^2\beta.$$

This agrees absolutely with LÉVY'S solution.

(*b.*) For the rectangular-shaped dam, remembering that the shear vanishes at the front (and at the flank),

$$\widehat{zx} = -\frac{3z^2}{\tau}\frac{x}{\tau}\left(1 - \frac{x}{\tau}\right).$$

LÉVY has the additional term

$$-\left\{-\frac{\tau}{5} + x\left(1 - \frac{x}{\tau}\right)\right\}\frac{x}{\tau}\left(1 - \frac{x}{\tau}\right).$$

The ratio of this additional term to the term above is

$$-\tfrac{1}{5} + \frac{x}{\tau}\left(1 - \frac{x}{\tau}\right) \quad \text{to} \quad \frac{3z^2}{\tau^2}.$$

The maximum value $= 20\tau^2/(3z^2)$ to 100, and occurs at the front and flank; for a dam double as high as broad it is 1·7 per cent. At quarter sections the ratio is $1{\cdot}25\,\tau^2/(3z^2)$ to 100, and at mid-section $5\tau^2/(3z^2)$ to 100; or for the same dimensions, well under 1 per cent. Thus the maximum deviation occurs at the terminals of the section just where the shear is zero. We may conclude, therefore, that for a dam of rectangular section the symmetrical parabolic distribution of shear over the base is as close a distribution as we can find in our ignorance of the nature of the real reactionary stresses of the substratum. It might seem accordingly desirable to treat the triangular-sectioned dam by a linear shear distribution across the base, and the rectangular-sectioned by a parabolic distribution. Actual dams are, however, neither of these shapes. Even if a triangular-shaped dam were constructed, the tail would be certain in practice to be truncated, and the base shear, supposing no batter of the front, would then vanish at both front and flank. It could not possibly, therefore, be given by a straight line. The following construction may then be used for the case of the usual dam with truncated tail:—

Turning to the second equation of (v.), p. 7, we have

$$\widehat{xz} = -\int_0^{\xi}\left(\gamma + \frac{d\widehat{zz}}{dz}\right)d\xi,$$

ξ being measured from the truncated tail, so that $\delta\xi = \delta x$.

Hence

$$\widehat{xz} = -\frac{d}{dz}\int_0^{\xi}(\gamma z + \widehat{zz})\,\delta\xi.$$

Now take two sections of the dam at distance ϵ apart, one the base section and one a small distance from it, and let the normal stresses on these at distance ξ from the tail be $\widehat{zz}_1$ and $\widehat{zz}_2$, and be given by

$$\widehat{zz}_1 = m_1\xi - c_1 \qquad \widehat{zz}_2 = m_2\xi - c_2,$$

we find

$$\widehat{xz} = -\gamma\xi - \frac{m_1 - m_2}{2\epsilon}\xi^2 + \frac{c_1 - c_2}{\epsilon}\xi.$$

Thus the distribution of shear on the base is undoubtedly parabolic if the normal pressures are taken to be linear, unless $m_1 = m_2$, which is actually the case in LÉVY'S triangle solution.

If the front of the dam be only very slightly battered, or takes a vertical step before the batter begins, then $\widehat{xz}$ must vanish at the front as well as the flank, and we conclude that the shear on the base will be given by the symmetrical parabola of equation (iv.) of p. 5.

This is the justification of the statement made that the parabolic distribution of shear is in practice as justifiable as the linear distribution of pressure.

In the case of the dam dealt with in the present paper, the blocking up of the tail has not been considered. Taking it into account, we find that the parabola of shear above indicated is, if height of masonry be the scale,

$$\text{Shear} = 1{\cdot}941\xi - {\cdot}027\xi^2$$

approximately. This has its axis *only one foot* removed from the centre of the base (73·36 feet base length). The parabola of shear given by (iv.) is

$$\text{Shear} = 2{\cdot}485\xi - {\cdot}034\xi^2.$$

Considering the approximate character of the whole treatment, we think the symmetrical distribution of shear is as likely to express the facts of the actual distribution as the earlier parabola. We are quite clear that to treat the dam as approximately triangular and use LÉVY'S line with a maximum shear at the flank would not in the least fit the case. It is obvious, however, that slight modifications of the contour of front or flank at the base theoretically may very considerably change the base distribution of shear.

Plate II.

A GRAPHICAL TREATMENT OF THE STABILITY OF DAMS.

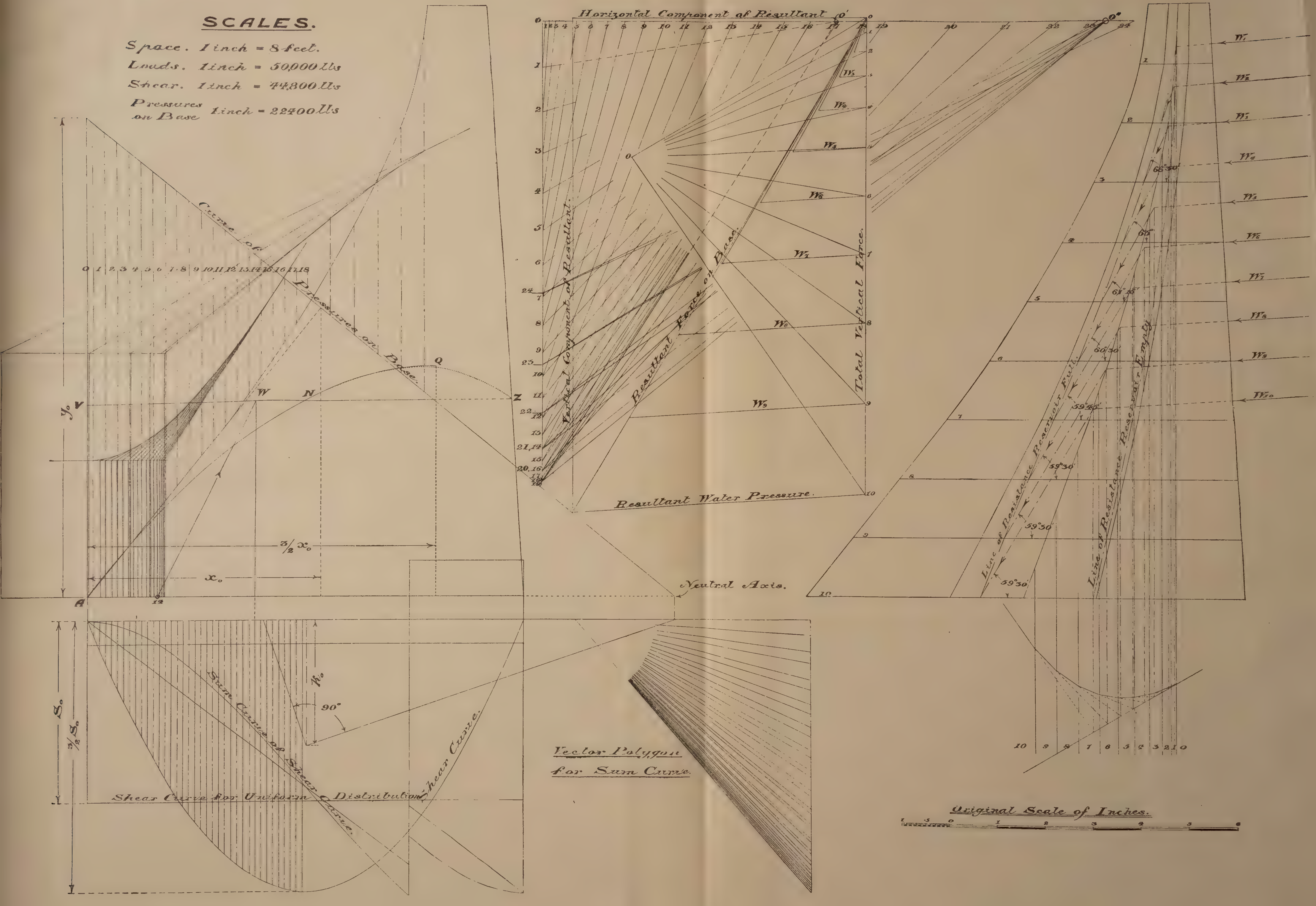
SCALES.

Space. 1 inch = 8 feet.

Loads. 1 inch = 50,000 lbs

Shear. 1 inch = 44,800 lbs

Pressures on Base 1 inch = 22400 lbs



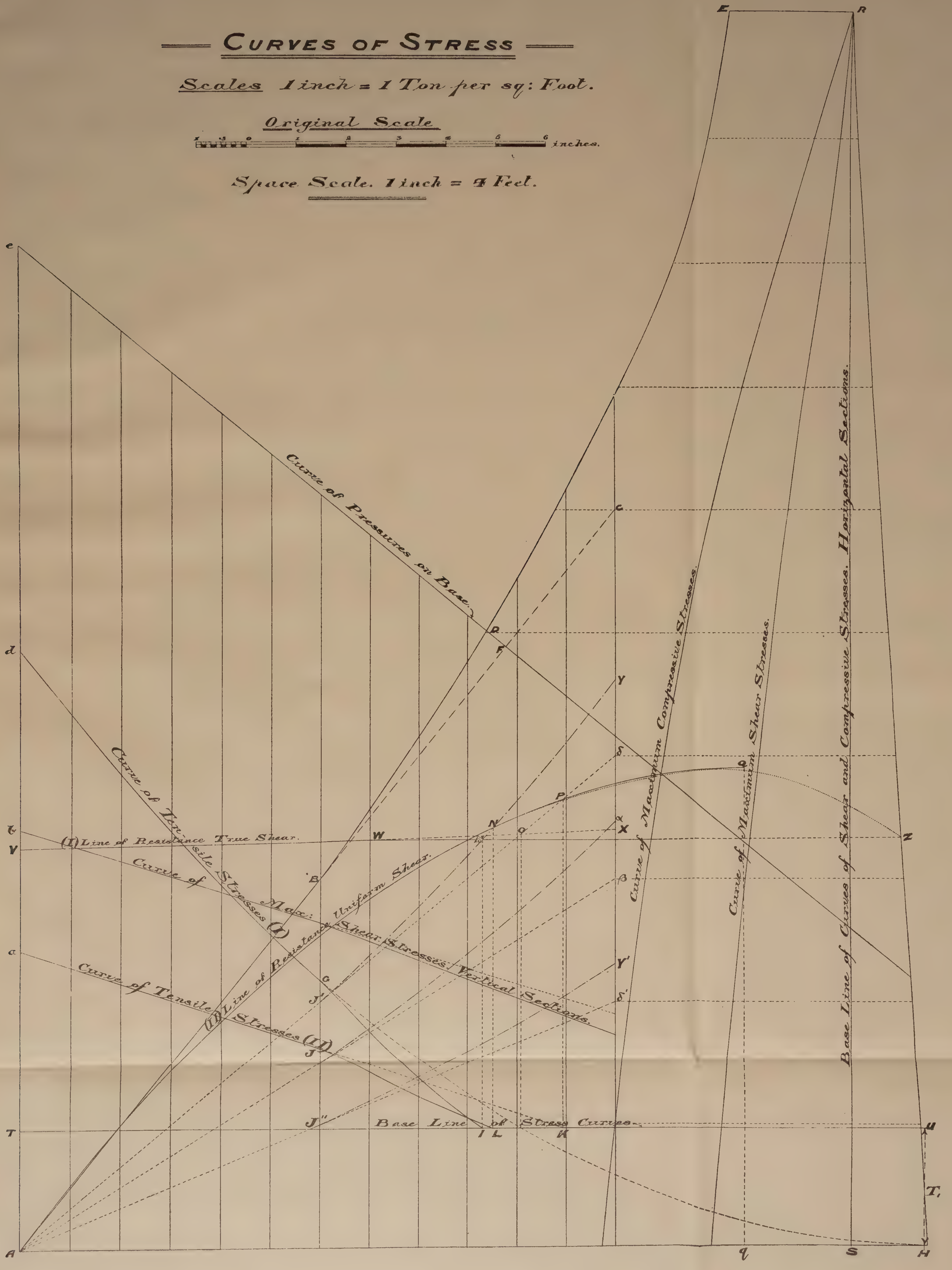
A GRAPHICAL TREATMENT OF THE STABILITY OF DAMS

Curves of Stress

Scales 1 inch = 1 Ton per sq: Foot.

Original Scale

Space Scale. 1 inch = 4 Feet.


1
·5
0
1
2
3
4
5
6 inches.
E
R
e
d
b
V
a
T
A
Curve of Pressures on Base.
D
F
C
Y
δ
2
X
β
Y'
δ'
N
O
P
Q
Z
W
B
G
J'
J
J''
(I) Line of Resistance True Shear.
Curve of Tensile Stresses (I)
Curve of Max: Shear Stresses. Vertical Sections.
(II) Line of Resistance Uniform Shear.
Curve of Tensile Stresses (II)
Base Line of Stress Curves.
I
L
K
U
T,
q
S
H
Curve of Maximum Compressive Stresses.
Curve of Maximum Shear Stresses.
Base Line of Curves of Shear and Compressive Stresses. Horizontal Sections.

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